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Vector optimization problems with quasiconvex constraints

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Abstract Let X be a real linear space, $X_0 \subset X$ a convex set, Y and Z topological real linear spaces. The constrained optimization problem $\min_C f(x)$, $g(x) \in -K$ is considered, where $f : X_0 \to Y$ and $g : X_0 \to Z$ are given (nonsmooth) functions, and $C \subset Y$ and $K \subset Z$ are closed convex cones. The weakly efficient solutions (w-minimizers) of this problem are investigated. When g obeys quasiconvex properties, first-order necessary and first-order sufficient optimality conditions in terms of Dini directional derivatives are obtained. In the special case of problems with pseudoconvex data it is shown that these conditions characterize the global w-minimizers and generalize known results from convex vector programming. The obtained results are applied to the special case of problems with finite dimensional image spaces and ordering cones the positive orthants, in particular to scalar problems with quasiconvex constraints. It is shown, that the quasiconvexity of the constraints allows to formulate the optimality conditions using the more simple single valued Dini derivatives instead of the set valued ones.

Keywords Vector optimization · Nonsmooth optimization · Quasiconvex vector functions · Pseudoconvex vector functions · Dini derivatives · Quasiconvex programming · Kuhn-Tucker conditions

Mathematics Subject Classifications 2000 90C46 · 90C26 · 26B25 · 49J52

1 Introduction

In this paper X is a linear space, $X_0 \subset X$ is a convex set, and Y and Z are topological linear spaces (tls). We deal only with real spaces. We consider the constrained vector optimization problems

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$$\min_C f(x), \quad g(x) \in -K, \tag{1}$$

where $f : X_0 \to Y$ and $g : X_0 \to Z$ are given functions, and $C \subset Y$ and $K \subset Z$ are closed convex cones. Confining to problems for which g obeys quasiconvex properties, and dealing with weakly efficient solutions (w-minimizers), we obtain first-order necessary and first-order sufficient optimality conditions in terms of Dini directional derivatives. Optimality conditions in terms of Dini set valued directional derivatives for problems with locally Lipschitz data have been investigated in [13]. Since here we deal in general with non-Lipschitz problems, to adopt a similar approach we introduce infinite elements in the image spaces. We show that the quasiconvexity of the constraints allows in important cases to substitute the set valued Dini derivative with the more simple single valued lower Dini derivative. A special care is paid for problems with pseudoconvex data. For such problems we obtain a characterization of the global w-minimizers and recognize that the obtained results generalize known ones for problems with convex data. Let us underline that similar generalizations to smooth scalar problems with quasiconvex constraints have given the origin of quasiconvex programming, see e. g. [2], [18] or [19]. This has given us an inspiration for the present study. Within this framework it can be considered as an attempt to generalize some basic results of quasiconvex programming from scalar to vector problems on one hand and from smooth to nonsmooth problems on the other hand. Concrete classical results, for instance ones in [2–4,15] show some similarities with the results of the present paper. A detailed comparison demands further considerations accounting also the different approaches. The main difference is that the present study is based on directional derivatives. In consequence the multipliers in the Lagrangian are directionally dependent. We find this approach more sensitive to treat nonsmooth and vector problems (recall that even smooth vector problems obey nonsmooth scalarization [14], in this sense we claim that all vector problems show nonsmooth behaviour). Example 8.4 gives a support to such a sentence. For the given there problem with quasiconvex data the sufficient conditions with directionally dependent multipliers work, while similar conditions with directionally independent multipliers fail. Another feature of the paper is the usage of extensions with infinite elements of the image spaces introduced in Sect. 3 and different than the one or two points extensions used sometimes in vector optimization, say in [1,7,11,12]. We find that often such extensions could be more convenient in vector optimization than these with one or two points.

2 Concepts of optimality

Let $x^0 \in X_0$. We put $X_0(x^0) = \{u \in X \mid x^0 + tu \in X_0 \text{ for some } t > 0\}$. The elements of $X_0(x^0)$ are called *admissible directions* for X_0 at x^0.

The point x^0 is said *feasible* for problem (1) if $g(x^0) \in -K$. In the sequel we use the following concepts of optimality.

Definition 2.1 We say that x^0 is a radial w-minimizer of problem (1) if $x^0 \in X_0$, x^0 is a feasible point, and for any $u \in X_0(x^0)$ there exists $\delta(u) > 0$ such that $f(x^0 + tu) - f(x^0) \notin -\operatorname{int} C$ whenever $0 < t < \delta(u)$, $x^0 + tu \in X_0$ and $x^0 + tu$ is feasible.

Definition 2.2 We say that x^0 is a global w-minimizer of problem (1) if $x^0 \in X_0$, x^0 is a feasible point and $f(x) - f(x^0) \notin -\operatorname{int} C$ for all feasible points $x \in X_0$.

Definition 2.3 The global w-minimizer x^0 is called strict if $f(x) - f(x^0) \notin -C$ for all feasible points $x \in X_0 \backslash \{x^0\}$.

Dealing with problems with scalar objective function, that is when $Y = \mathbb{R}$ and $C = \mathbb{R}_+$, we use to say just minimizers instead of w-minimizers.

Obviously, each strict global w-minimizer is a global w-minimizer, and each global w-minimizer is a radial w-minimizer.

The well known definition of a local w-minimizers (weakly efficient point) applies for the case when X is a tls. The feasible point $x^0 \in X_0$ is said a local w-minimizers of problem (1) if there is a neighbourhood U of x^0 such that $f(x) - f(x^0) \notin -\text{int}\, C$ for all feasible points $x \in X_0 \cap U$. Obviously, then each global w-minimizer is a local w-minimizer, and each local w-minimizer is a radial w-minimizer. Due to this observations, the necessary conditions for a radial w-minimizer are also necessary conditions for a local w-minimizer, and the sufficient conditions for a global w-minimizer are also sufficient conditions for a local w-minimizer. Since only optimality conditions of these types are considered in the sequel, the eventual discussion on local w-minimizers is omitted. We find the notion of a radial w-minimizer convenient when treating optimization problems through directional derivatives. Besides, we gain the advantage to consider problems in which a topological structure of the linear space X is not assumed.

When $\text{int}\, C = \emptyset$ then straightforward from the definitions each feasible point $x^0 \in X_0$ is a radial w-minimizer and a global w-minimizer. Therefore the interesting case is when C has a nonempty interior.

In Definition 2.1 the notion of a radial minimizer is introduced. Generally, we say that certain radial property holds at a point x^0 if the property is satisfied along the rays starting at x^0. Besides the radial minimisers, we will use also the notion of radial continuity. Let T be a tls. We say that the function $\phi : X_0 \to T$ is *radially continuous* at $x^0 \in X_0$ if for any $u \in X_0(x^0)$ the function $t \to \phi(x^0 + tu)$, $t \geq 0$ (such that $x^0 + tu \in X_0$), is continuous at $t^0 = 0$. The function ϕ is said radially continuous if it is radially continuous at any $x^0 \in X_0$.

3 Extension of linear spaces with infinite elements

Let T be a linear space. We can extend T with infinite elements. To any $v \in T\backslash\{0\}$ we juxtapose the infinite element v_∞, and accept that $v^1_\infty = v^2_\infty$ if and only if $v^2 = \lambda v^1$ for some $\lambda > 0$. Denote by T_∞ the set of the infinite elements, then we will consider the extension $\overline{T} = T \cup T_\infty$.

When T is a tls, then a topology in $\overline{T}$ can be introduced in terms of local bases of neighbourhoods. If $v \in T$ and $\mathcal{B}(v)$ is a local base of neighbourhoods of v in T, we accept that $\mathcal{B}(v)$ is also a local base of neighbourhoods in $\overline{T}$. If v_∞ is the infinite element corresponding to $v \in T\backslash\{0\}$, then the family $\mathcal{B}(v_\infty) = \{(t + W) \cup W_\infty\}$, where $W \subset T$ is an arbitrary open convex cone such that $v \in W$ and t is an arbitrary point in T, constitutes a local base of neighbourhoods of v_∞. Saying that W is an open cone we mean that W is an open set in T such that $\lambda W \subset W$ for all $\lambda > 0$. For a cone W we write $W_\infty = \{w_\infty \mid w \in W\backslash\{0\}\}$. Further we use also the notation $\overline{W} = W \cup W_\infty$. To prove that the intersection of two sets in $\mathcal{B}(v_\infty)$ contains an element of $\mathcal{B}(v_\infty)$ one should observe, that when $t_1, t_2 \in T$ and W_1, W_2 are open convex cones containing v, then there exists $\lambda_0 > 0$, such that $\lambda_0 v + W_1 \cap W_2 \subset t_i + W_1 \cap W_2$, $i = 1, 2$. Indeed, take $\lambda_0 > 0$ such that $v - t_i/\lambda_0 \in W_1 \cap W_2$. Now $\lambda_0 v \in t_i + \lambda_0 W_1 \cap W_2 = t_i + W_1 \cap W_2$ and $\lambda_0 v + W_1 \cap W_2 \subset t_i + W_1 \cap W_2 + W_1 \cap W_2 = t_i + W_1 \cap W_2$.

Theorem 3.1 *If T is finite dimensional, then the extension $\overline{T}$ is compact.*

Proof We consider T with its Euclidean metrics. Take an open covering $\bar{\mathcal{G}} = \mathcal{G} \cup \mathcal{G}_\infty$ of $\overline{T}$ with sets from the described above local bases, where $\mathcal{G}$ consist of bounded open sets and $\mathcal{G}_\infty$ of sets of the type $(t + W) \cup W_\infty$ with $t \in T$ and W an open convex cone. Since the family $\mathcal{W}_\infty = \{W_\infty \mid (t + W) \cup W_\infty \in \mathcal{G}_\infty$ for some $t \in T\}$ covers T_∞, the family $\mathcal{W} = \{W \mid W_\infty \in \mathcal{W}_\infty\}$ covers the unit sphere $S = \{t \in T \mid \|t\| = 1\}$. Since S is compact, the set S is covered by a finite subfamily $\mathcal{W}^0 = \{W^i\}_{i=1}^k \subset \mathcal{W}$. It is clear, that the corresponding finite family $\mathcal{G}_\infty^0 = \{(t^i + W^i) \cup W_\infty^i\}_{i=1}^k$ covers T_∞. The set $T^0 = T \setminus \bigcup_{i=1}^k (t^i + W^i)$ is compact. The closedness of T^0 is evident. The boundedness of T^0 is shown by the following reasoning. Take $\delta_0 = \min_{x \in S} \max_{1 \le i \le k} \operatorname{dist}(x, T \setminus W^i)$ (here $\operatorname{dist}(\cdot, \cdot)$ is the point-to-set distance). Then $\delta_0 > 0$ because the distance function is continuous, S is compact and $\max_{1 \le i \le k} \operatorname{dist}(x, T \setminus W^i) > 0$ for $x \in S$, the latter is a consequence of $x \in W^i$ for at least one i. Take δ such that $0 < \delta < \delta_0$. Let $u \in S$. Consider the cone $L = \{t \in T \mid \|t/\lambda - u\| \le \delta$ for some $\lambda > 0\}$. Due to the definition of δ, there exists an index i such that $L \subset W^i$. Let $\lambda_0 = (1/\delta) \max_{1 \le j \le k} \|t_i\|$. When $\lambda \ge \lambda_0$ we have $\|(u - t^i/\lambda) - u\| = \|t^i\|/\lambda \le \delta$. Therefore $u - t^i/\lambda \in W^i$ and $\lambda u \in t^i + W^i \subset T \setminus T^0$. This reasoning shows that, when a point $\lambda u \in T^0$ then $\lambda < \lambda_0$, hence T^0 is bounded. Thus, the set T^0 is compact and has an open covering $\mathcal{G}$. Therefore it can be covered by a finite subfamily $\mathcal{G}^0$ of $\mathcal{G}$. In consequence $\overline{T}$ is covered by the finite subfamily $\bar{\mathcal{G}}^0 = \mathcal{G}^0 \cup \mathcal{G}_\infty^0$ which shows that $\overline{T}$ is compact. □

Since $\overline{T}$ is a topological space, we can apply topological operations in $\overline{T}$. In particular the interior and the closure in $\overline{T}$ are denoted respectively $\overline{\text{int}}$ and $\overline{\text{cl}}$. The overline is put to distinguish from the interior and the closure in T which are denoted respectively int and cl. Since $T \subset \overline{T}$, the topological operation in T, in particular the operations int and cl, can be considered also as operations in $\overline{T}$. We adopt different notations for the operations in T and $\overline{T}$, since applied to sets in $\overline{T}$ they give in general different results. For instance, if T is a locally convex space and W is a closed cone in T we have $\operatorname{cl} W = W$ while $\overline{\operatorname{cl}}\, W = W \cup W_\infty$. We explain the latter equality concentrating on the infinite points. To explain the inclusion $\overline{\operatorname{cl}}\, W \supset W \cup W_\infty$ (true also when T is arbitrary tls and not necessarily locally convex space) let v_∞ be an infinite point corresponding to the point $v \in W \setminus \{0\}$. Take a neighbourhood $(t + W^0) \cup W_\infty^0$ of v_∞ where W^0 is a convex open cone containing v. Since W^0 is open and contains v, we have $v - t/\lambda \in W^0$ for some $\lambda > 0$, whence $\lambda v \in (t + W^0) \cap W$. This shows that any neighbourhood of v_∞ intersects W, hence $v_\infty \in \overline{\operatorname{cl}}\, W$. To explain the inclusion $\overline{\operatorname{cl}}\, W \subset W \cup W_\infty$ let $v_\infty \notin W_\infty$ be an infinite point corresponding to $v \in T \setminus \{0\}$. As a consequence of $v_\infty \notin W_\infty$ we have $v \notin W$. Since T is locally convex space, there exists an open convex cone W^0 containing v and not intersecting W. Then $W^0 \cup W_\infty^0$ is a neighbourhood of v_∞ which does not intersect W, hence $v_\infty \notin \overline{\operatorname{cl}}\, W$.

The following property plays an important role in the proof of some of the forthcoming results: *If $W \subset T$ is a cone, then* $\overline{\operatorname{int}}\, \overline{W} \cap T = \operatorname{int} W$ *(the same is true when $\overline{W}$ is arbitrary set in $\overline{T}$ and $W = \overline{W} \cap T$)*. Indeed, from the definition of the topology in $\overline{T}$, the finite point v belongs to $\overline{\operatorname{int}}\, \overline{W}$ if and only if we have $v \subset U \subset \operatorname{int} W$ for some neighbourhood U of v, that is if $v \in \operatorname{int} W$. *Similarly, when T is locally convex space and $W \subset T$ is a cone, we have* $\overline{\operatorname{int}}\, \overline{W} = \operatorname{int} W \cup (\operatorname{int} W)_\infty$. To get the proof it remains to consider the infinite points. Let $v_\infty \in \overline{\operatorname{int}}\, \overline{W}$ is an infinite point corresponding to $v \in T \setminus \{0\}$. Then there exists an open convex cone W^0 such that $v \in W^0$ and for some $t \in T$ we have $W_\infty^0 = ((t + W^0) \cup W_\infty^0) \cap T_\infty \subset W_\infty$. This implies $W^0 \subset W$ and with regard to W^0 open it holds $v \in \operatorname{int} W$, consequently $v_\infty \in (\operatorname{int} W)_\infty$ and $\overline{\operatorname{int}}\, \overline{W} \subset \operatorname{int} W \cup (\operatorname{int} W)_\infty$ (true for arbitrary tls T). To prove the converse inclusion, take the infinite point $v_\infty \in (\operatorname{int} W)_\infty$ corresponding to $v \in \operatorname{int} W$. Then there exists an open convex cone W^0 (here the local convexity of T is used) such that $v_\infty \in W_\infty^0 \subset \overline{\operatorname{int}}\, \overline{W}$.

When $T = \mathbb{R}$ we have $\overline{T} = \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$. Attention, for the integer $k > 1$ one should not mix $\overline{\mathbb{R}^k}$ (the extension of $\mathbb{R}^k$ with infinite points) and $\overline{\mathbb{R}}^k$ (the Cartesian product $\overline{\mathbb{R}} \times \cdots \times \overline{\mathbb{R}}$ of k copies of $\overline{\mathbb{R}}$). It holds $\mathbb{R}^k \subset \overline{\mathbb{R}^k}$ and $\mathbb{R}^k \subset \overline{\mathbb{R}}^k = (\overline{\mathbb{R}})^k$ but neither of the sets $\overline{\mathbb{R}^k}$ and $\overline{\mathbb{R}}^k$ is contained in the other one. In the sequel, if T_1 and T_2 are tls, and $A \subset \overline{T}_1$ and $B \subset \overline{T}_2$, we understand the interior $\overline{\text{int}}\ A \times B = \overline{\text{int}}\ A \times \overline{\text{int}}\ B$ as interior in the product space $\overline{T}_1 \times \overline{T}_2$ (and not in $\overline{T_1 \times T_2}$, turn attention that $A \times B$ need not be contained in $\overline{T_1 \times T_2}$).

Let T be a tls and $\phi : X_0 \to T$ a given function. Take $x \in X_0$ and $u \in X_0(x)$. The Kuratowski limit $\text{Liminf}_{t \to 0^+} \frac{1}{t}(\phi(x + tu) - \phi(x))$ when considered in T is denoted by $\phi_-^{(1)}(x, u)$, and when considered in $\overline{T}$ by $\phi_-^{[1]}(x, u)$. In both cases we use to say that this limit is the *(set valued) Dini derivative* of ϕ at x in direction u. Pay attention, that due to Theorem 3.1 when T is finite dimensional it holds $\phi_-^{[1]}(x, u) \neq \emptyset$.

For a scalar function $\phi : X_0 \to \mathbb{R}$ we will apply also the single valued lower Dini derivative defined as $\phi'_-(x, u) = \liminf_{t \to 0^+} \frac{1}{t}(\phi(x + tu) - \phi(x))$, whose values are in $\overline{\mathbb{R}}$. The equality $\phi'_-(x, u) = \inf \phi_-^{[1]}(x, u)$ (the infimum taken in $\overline{\mathbb{R}}$) relates the single valued and the set valued Dini derivatives. For a vector function $\phi = (\phi_1, \ldots, \phi_k) : X_0 \to \mathbb{R}^k$ we define the *single valued lower Dini derivative* by $\phi'_-(x, u) = ({\phi_1}'_-(x, u), \ldots, {\phi_k}'_-(x, u)) \in \overline{\mathbb{R}}^k$.

The extension $\overline{\mathbb{R}}$ of the real number set $\mathbb{R}$ with the infinite points $\pm\infty$ is widely used in convex and in nonsmooth scalar optimization, and to some extend in vector optimization. The proposed here extension is applied in [9] to study vector variational inequalities. Theorem 3.1 appears there but without a rigorous proof, so this gap is fulfilled here. A different approach is given in [12] where a two-point extension with infinite points of a linear space T partially ordered by a cone W is proposed. Recently Durea [11] uses the same two-point extensions studying Lagrange claims for set valued optimization. His motivation is to cover in a unified theory both cases of a set valued optimization and of real extended functions. The two point extension is $\overline{T}_W = T \cup \{-\infty_W\} \cup \{+\infty_W\}$ and when W is a convex cone with nonempty interior can be represented through the defined here extension $\overline{T}$ by $\overline{T}_W = T \cup \{-w_\infty\} \cup \{w_\infty\}$ where $w \in \text{int}\ W$. This point of view allows using the heritage topology to introduce straightforward a topology in $\overline{T}_W$ (though a direct definition is simple enough). Similarly, possibly defined in advance algebraic and cone-ordered structures in $\overline{T}$ (we do not follow this line here, for this exceeds the aim of the paper) can be inherited by $\overline{T}_W$. Let us underline, that in vector optimization one or two-point extensions with infinite elements of cone-ordered linear spaces use also other authors, see e. g. Ref. [7] and [1].

In our opinion, the proposed here extension, though looking more complex, has some advantage. It does not refer to the ordering cone. Because of this the set valued Dini derivative $\phi_-^{[1]}(x, u)$ can be defined in tls, which need not be ordered by a cone (it is not in the nature of the concept to associate $\phi_-^{[1]}(x, u)$ with an ordering cone). We find the extension $\overline{T}$ appropriate for the forthcoming discussion. Finally, it is to some extend similar to the way, in which in projective geometry infinite elements are defined.

When T is a tls, then T^* denotes the dual space of T, and $\langle \cdot, \cdot \rangle$ the dual pairing on $T^* \times T$. Recall that when T is a normed space, then T^* is a Banach space.

When T is a tls, we extend the values of the continuous linear functionals $\xi \in T^*$ on the infinite elements in T_∞, putting $\langle \xi, v_\infty \rangle = +\infty$ when $\langle \xi, v \rangle > 0$, $\langle \xi, v_\infty \rangle = 0$ when $\langle \xi, v \rangle = 0$, and $\langle \xi, v_\infty \rangle = -\infty$ when $\langle \xi, v \rangle < 0$ (here $v_\infty \in T_\infty$ is the infinite point corresponding to the finite one $v \in T \backslash \{0\}$).

4 Concepts of quasiconvexity and pseudoconvexity

In this section like in the previous one T denotes a tls, and W a closed convex cone in T. Recall that the positive polar cone of W is the cone $W' = \{\xi \in T^* \mid \langle \xi, w\rangle \geq 0 \text{ for all } w \in W\}$ and $W'' = (W')' = \{w \in T \mid \langle \xi, w\rangle \geq 0 \text{ for all } \xi \in W'\}$ is the second polar cone. From the definition of W' we have $W \subset W''$, and when T is locally convex due to the Separation Theorem [21] we have $W'' = W$ ($W'' = \operatorname{cl}\operatorname{co} W$ for arbitrary cone W). Possible use of this observation is the following: to show that a point $x \in X_0$ is feasible for problem (1) it is enough to show that $g(x) \in -K''$ (further this is used e. g. in the proofs of Theorems 5.1 and 5.3).

When $w^0 \in W$ we put $W'[w^0] = \{\xi \in W' \mid \langle \xi, w^0\rangle = 0\}$ and $W[w^0] = (W'[w^0])' = \{w \in T \mid \langle \xi, w\rangle \geq 0 \text{ when } \xi \in W',\ \langle \xi, w^0\rangle = 0\}$. The equality $\langle \xi, w^0\rangle = 0$ usually referred as complementary slackness condition enters into the definition of both $W'[w^0]$ and $W[w^0]$. It can be shown that $W[w^0]$ is the tangent cone of W at w^0.

Recall that $\xi \in T^*$ is called an extreme direction for W' if $\xi \in W'\backslash\{0\}$, and for all $\xi^1, \xi^2 \in W'$, such that $\xi = \xi^1 + \xi^2$, it holds $\xi^1, \xi^2 \in \mathbb{R}_+ \xi$. The set of the extreme directions of W' is denoted $\operatorname{extd} W'$. If T is locally convex space and W has a weak*-compact base, due to Krein-Milman Theorem [21] we have $W = \{w \in T \mid \langle \xi, w\rangle \geq 0,\ \xi \in \operatorname{extd} W'\}$.

Theorem 4.1 *Let $w^0 \in W$. Then the equality* $\operatorname{extd} W'[w^0] = \{\xi \in \operatorname{extd} W' \mid \langle \xi, w^0\rangle = 0\}$ *relates the extreme directions of $W'[w^0]$ and W'.*

Proof The crucial moment is to show that if $\xi \in \operatorname{extd} W'[w^0]$ then $\xi \in \operatorname{extd} W'$ (the opposite inclusion is obvious). Let $\xi = \xi^1 + \xi^2$ with $\xi^i \in W'$ $(i = 1, 2)$. Then $0 = \langle \xi, w^0\rangle = \langle \xi^1, w^0\rangle + \langle \xi^2, w^0\rangle$. Since $\langle \xi^i, w^0\rangle \geq 0$ $(i = 1, 2)$, we get the equalities $\langle \xi^i, w^0\rangle = 0$ $(i = 1, 2)$. So $\xi^i \in W'[w^0]$ and since $\xi \in \operatorname{extd} W'[w^0]$, we get $\xi^i \in \mathbb{R}_+\xi$ $(i = 1, 2)$. □

For the function $\phi : X_0 \to T$ the level set corresponding to $t \in T$ is the set $\operatorname{lev}_{t,W}\phi = \{x \in X_0 \mid \phi(x) \in t - W\}$.

Definition 4.1 **([17])** The function $\phi : X_0 \to T$ is said W-quasiconvex if for all $t \in T$ the level set $\operatorname{lev}_{t,W}\phi$ is convex. In other words, the function ϕ is W-quasiconvex if for all $t \in T$, all $x^1, x^2 \in X_0$, $x^1 \neq x^2$, such that $\phi(x^1) \in t - W$, $\phi(x^2) \in t - W$, and all $\tau \in (0, 1)$, it holds $\phi((1-\tau)x^2 + \tau x^1) \in t - W$.

The next theorem gives a characterization of W-quasiconvex functions when T is a Banach space and $\operatorname{int} W \neq \emptyset$. Moreover, it characterizes W-quasiconvex functions when W introduces a directed order on T. The latter means that for all $t^1, t^2 \in T$ there exists $t \in T$ such that $t - t^i \in W$, $i = 1, 2$. Observe that if $\operatorname{int} W \neq \emptyset$ then W introduces a directed order on T. We write $\operatorname{cl}^* W'$ for the weak* closure of W'. Let us note, that if $\operatorname{int} W \neq \emptyset$ then W' has a bounded hence weak* compact base and by the Krein-Milman Theorem [21] the hypothesis $W' = \operatorname{cl}^* \operatorname{co} \operatorname{extd} W'$ is fulfilled.

Theorem 4.2 (Benoist et al. [5]) *Let T be a Banach space. Assume that W introduces a directed order on T and* $W' = \operatorname{cl}^* \operatorname{co} \operatorname{extd} W'$. *Then $\phi : X_0 \to T$ is W-quasiconvex if and only if the functions $\langle \xi, \phi\rangle$ are quasiconvex for all $\xi \in \operatorname{extd} W'$.*

The following theorem in the case of a polyhedral cone W' extends Theorem 4.2 to locally convex spaces. It generalizes also the result of Luc [17,Proposition 6.5, p. 30] which concerns the particular case when T is the Euclidean space $\mathbb{R}^n$ and W is generated by exactly n linearly independent vectors.

Theorem 4.3 *Let T be a locally convex space and let the cone W' be polyhedral. Then the function $\phi : X_0 \to T$ is W-quasiconvex if and only if the functions $\langle \xi, \phi \rangle$ are quasiconvex for all $\xi \in \operatorname{extd} W'$.*

Proof If W' is polyhedral, then it admits a base $\Gamma = \operatorname{co}\{\xi^1, \dots \xi^k\}$ where $\{\xi^1, \dots \xi^k\} = \Gamma \cap \operatorname{extd} W'$. It is enough to prove that ϕ is W-quasiconvex if and only if the functions $\langle \xi^i, \phi \rangle$ are quasiconvex for all $i = 1, \dots, k$. Suppose that the vectors $\{\xi^1, \dots, \xi^n\}$ are linearly independent, while $\xi^j = \sum_{i=1}^n \lambda_i^j \xi^i$ for $j = n+1, \dots, k$. We put into correspondence of ϕ the function $\phi^0 : X_0 \to \mathbb{R}^n$, $\phi^0(x) = (\langle \xi^1, \phi(x) \rangle, \dots, \langle \xi^n, \phi(x) \rangle)$. Put $e^i = (0, \dots, 1, \dots, 0)$, $i = 1, \dots, n$ (the only unit is on i-th place). Let $W^0 \subset \mathbb{R}^n$ (with the Euclidean norm) be the positive polar cone of the convex cone $(W^0)'$ generated by the vectors e^i, $i = 1, \dots, n$, and $e^j = \sum_{i=1}^n \lambda_i^j e^i$, $j = n+1, \dots, k$. Then the function ϕ is W-quasiconvex if and only if the function ϕ^0 is W^0-quasiconvex. Indeed, let ϕ^0 be W^0-quasiconvex. Take a point $t \in T$ and let $t^0 = (\langle \xi^1, t \rangle, \dots, \langle \xi^n, t \rangle) \in \mathbb{R}^n$. From the W^0-quasiconvexity of ϕ^0 the set $L_t = \{x \in X_0 \mid \phi^0(x) \in t^0 - W^0\}$ is convex. An easy calculation shows that $L_t = \{x \in X_0 \mid \phi(x) \in t - W\}$, whence ϕ is W-quasiconvex. To verify $L_t = \{x \in X_0 \mid \phi(x) \in t - W\}$ we observe (applying $W = W''$, true because T is locally convex space) that $x \in L_t$ if and only if it holds both:

$$\left\langle e^\imath, \sum_{i=1}^n \langle \xi^i, \phi(x) - t \rangle\, e^i \right\rangle = \langle \xi^i, \phi(x) - t \rangle \le 0 \quad \text{for} \quad \imath = 1, \dots, n\,;$$

$$\begin{aligned}\left\langle e^j, \sum_{i=1}^n \langle \xi^i, \phi(x) - t \rangle\, e^i \right\rangle &= \left\langle \sum_{i=1}^n \lambda_i^j\, e^i, \sum_{i=1}^n \langle \xi^i, \phi(x) - t \rangle\, e^i \right\rangle \\ &= \left\langle \sum_{i=1}^n \lambda_i^j \xi^i, \phi(x) - t \right\rangle \\ &= \langle \xi^j, \phi(x) - t \rangle \le 0 \quad \text{for} \quad j = n+1, \dots, k\,.\end{aligned}$$

Let now ϕ be W-quasiconvex. Take $t^0 \in \mathbb{R}^n$ and choose a point $t \in T$ to be a solution of the system of linear equations $\langle \xi^i, t \rangle = t_i^0$, $i = 1, \dots, n$. The resolvability of the system is a consequence of the linear independence of ξ^i, $i = 1, \dots, n$. Since ϕ is W-quasiconvex, the set $\{x \in X_0 \mid \phi(x) \in t - W\}$ is convex. Repeating the above calculations, we see that $\{x \in X_0 \mid \phi(x) \in t - W\} = \{x \in X_0 \mid \phi^0(x) \in t^0 - W^0\}$. This shows that ϕ^0 is W^0-quasiconvex. To complete the proof, it remains only to apply Theorem 4.2 for the function $\phi^0 : X_0 \to \mathbb{R}^n$ having an Euclidean (hence Banach) space as image space. □

The following theorem is in fact a corollary of Theorems 4.2 and 4.3.

Theorem 4.4 *Under the hypotheses of Theorems 4.2 or 4.3, if the function $\phi : X_0 \to T$ is W-quasiconvex and $w^0 \in W$, then ϕ is also $W[w^0]$-quasiconvex.*

Proof First of all we prove that the cone $W[w^0]$ also satisfies the hypotheses of Theorems 4.2 and 4.3 respectively.

Let, under the hypotheses of Theorems 4.2, W introduce a directed order on T and $W' = \operatorname{cl}^* \operatorname{co} \operatorname{extd} W'$. Let $t^1, t^2 \in T$ and $t \in T$ be such that $t - t^i \in W$, $i = 1, 2$. Since $W \subset W[w^0]$, we have also $t - t^i \in W[w^0]$, $i = 1, 2$. Therefore $W[w^0]$ introduces a directed order on T. Within the accepted notations $T'[w^0] = \{\xi \in T^* \mid \langle \xi, w^0 \rangle = 0\}$. Using the weak* closedness and the convexity of $T'[w^0]$, and applying Theorem 4.1, we get

$$\begin{aligned} W'[w^0] &= W' \cap T'[w^0] = \text{cl}^* \text{ co extd } W' \cap T'[w^0] \\ &= \text{cl}^* \left(\text{co extd } W' \cap T'[w^0]\right) = \text{cl}^*\text{co } \left(\text{extd } W' \cap T'[w^0]\right) \\ &= \text{cl}^* \text{ co extd } W'[w^0]. \end{aligned}$$

Let, under the hypotheses of Theorems 4.3, the cone W' be polyhedral. Then W' possesses a base Γ with $\Gamma \cap \text{extd } W'$ being a finite set. Then $\Gamma(w^0) = \Gamma \cap T'[w^0] \subset \Gamma$ is a base of $W'[w^0]$ and according to Theorem 4.1 extd $W'[w^0] = \text{extd } W' \cap T'[w^0] \subset \text{extd } W'$. Therefore the set $\Gamma(w^0) \cap \text{extd } W'[w^0] \subset \Gamma \cap \text{extd } W'$ is finite. Hence, the cone $W'[w^0]$ has a base containing finite number of extreme directions. Consequently, $W'[w^0]$ is polyhedral.

We prove now the thesis. Since ϕ is W-convex, according to Theorem 4.2 or 4.3 the functions $\langle \xi, \phi \rangle$, $\xi \in \text{extd } W'$, are quasiconvex. From Theorem 4.1 extd $W'[w^0] \subset \text{extd } W'$, whence the functions $\langle \xi, \phi \rangle$, $\xi \in \text{extd } W'[w^0]$, are quasiconvex. Then, according again to Theorem 4.2 or 4.3 the function ϕ is $W[w^0]$-quasiconvex. □

The quasiconvexity of the scalar function $\phi : X_0 \to \mathbb{R}$ can be defined also in the following way. We say that ϕ is quasiconvex if the inequality $\phi(x^2) \geq \phi(x^1)$, $x^1 \neq x^2$, and $0 < t < 1$ imply $\phi((1-t)x^2 + tx^1) \leq \phi(x^2)$. When in this definition we fix $x^2 = x^0$, then we will say that ϕ is quasiconvex at x^0 (the quasiconvexity at x^0 is a "radial notion" because it is a property which holds along the rays starting at x^0). The vector analogue of this definition is the following.

Definition 4.2 We say that the function $\phi : X_0 \to T$ is W-quasiconvex at $x^0 \in X_0$, if $\phi(x^1) - \phi(x^0) \in -W$, $x^1 \in X_0 \backslash \{x^0\}$, and $t \in (0,\ 1)$, imply $\phi((1-t)x^0 + tx^1) - \phi(x^0) \in -W$.

To characterize the W-quasiconvexity at x^0 of a given function $\phi : X_0 \to T$ through scalar functions we introduce the following definition of jointly quasiconvex at x^0 scalar functions $\phi^j : X_0 \to \mathbb{R}$, $j \in J$. Pay attention, that if all the functions ϕ^j, $j \in J$, are quasiconvex at x^0, then they are also jointly quasiconvex.

Definition 4.3 We say that the scalar functions $\phi^j : X_0 \to \mathbb{R}$, $j \in J$, are jointly quasiconvex at $x^0 \in X_0$, if when for a point $x^1 \in X_0 \backslash \{x^0\}$ all the inequalities $\phi^j(x^1) - \phi^j(x^0) \leq 0$, $j \in J$, are satisfied, and $t \in (0,\ 1)$, then also all the inequalities $\phi^j((1-t)x^0 + tx^1) - \phi^j(x^0) \leq 0$, $j \in J$, are satisfied.

Theorem 4.5 *Let T be a locally convex space. Then the function $\phi : X_0 \to T$ is W-quasiconvex at $x^0 \in X_0$ if and only if the functions $\langle \xi, \phi(x) \rangle$, $\xi \in W'$, are jointly quasiconvex at x^0. When W' has a weak* compact base Γ, then we can confine to the functions $\langle \xi, \phi(x) \rangle$, $\xi \in \Gamma \cap \text{extd } W'$.*

Proof The thesis is an obvious reformulation of the definition in terms of scalarization. The hypothesis that T is a locally convex space is assumed to guarantee $W = W''$. When W' has a weak* compact base the proof applies the Krein-Milman Theorem. □

In the sequel pseudoconvexity plays an important role. Recall that the function $\phi : X_0 \to \mathbb{R}$ is said pseudoconvex at $x^0 \in X_0$, if $\phi(x^0) > \phi(x^1)$, $x^1 \in X_0$, implies $\phi'_-(x^0, x^1 - x^0) < 0$. The function ϕ is said pseudoconvex, if it is pseudoconvex at each $x^0 \in X_0$. This definition of pseudoconvexity in terms of Dini derivatives is given by Diewert [10] as a convenient modification for nonsmooth functions of the classical definition of Mangasarian [19]. We generalize it to vector functions. Let us mention that when ϕ is directionally differentiable the pseudoconvexity from the following Definition 4.4 reduces to the one given by Cambini [6].

Definition 4.4 We say that $\phi : X_0 \to T$ is W-pseudoconvex at $x^0 \in X_0 \backslash \{x^0\}$, if $\phi(x^1) - \phi(x^0) \in -\text{int}\, W$, $x^1 \in X_0$, implies $\phi_-^{[1]}(x^0, x^1 - x^0) \cap -\overline{\text{int}}\ \overline{W} \neq \emptyset$. We say that ϕ is pseudoconvex, if it is pseudoconvex at each $x^0 \in X_0$.

Besides the pseudoconvexity, also the strict pseudoconvexity plays an important role in the sequel. We say that the function $\phi : X_0 \to \mathbb{R}$ is *strictly pseudoconvex at* $x^0 \in X_0$, if $\phi(x^0) \geq \phi(x^1)$, $x^1 \in X_0 \backslash \{x^0\}$, implies $\phi'_-(x^0, x^1 - x^0) < 0$. We say that ϕ is *strictly pseudoconvex*, if it is strictly pseudoconvex at each point $x^0 \in X_0$. Here is the vector analogue.

Definition 4.5 We say that $\phi : X_0 \to T$ is strictly W-pseudoconvex at $x^0 \in X_0$, if $\phi(x^1) - \phi(x^0) \in -W$, $x^1 \in X_0 \backslash \{x^0\}$, implies $\phi_-^{[1]}(x^0, x^1 - x^0) \cap -\overline{\text{int}}\ \overline{W} \neq \emptyset$. We say that ϕ is strictly pseudoconvex, if it is strictly pseudoconvex at each $x^0 \in X_0$.

Let the function $\phi : X_0 \to T$ be W-quasiconvex at $x^0 \in X_0$. Let $x^1 \in X_0 \backslash \{x^0\}$ be such that $\phi(x^1) - \phi(x^0) \in -W$, or equivalently $\phi(x^0 + u) - \phi(x^0) \in -W$ where $u = x^1 - x^0$. Now $\phi(x^0 + tu) - \phi(x^0) \in -W$ for $0 < t < 1$, whence $\phi_-^{[1]}(x^0, u) \subset -\overline{W}$. Hence $\phi_-^{[1]}(x^0, u) \cap -\overline{W} \neq \emptyset$ in the case when $\phi_-^{[1]}(x^0, u) \neq \emptyset$ (when T is finite dimensional this has place due to the compactness of $\overline{T}$). The definition of the strict W-pseudoconvexity at x^0 strengthens this property to $\phi_-^{[1]}(x^0, u) \cap -\overline{\text{int}}\,\overline{W} \neq \emptyset$. Nevertheless a strictly W-pseudoconvex at x^0 function need not be W-quasiconvex at x^0.

Example 4.1 Let $X = \mathbb{R}$, $X_0 = \mathbb{R}_+$, $x^0 = 0$, $T = \mathbb{R}$, $W = \mathbb{R}_+$. The function $\phi : X_0 \to T$, $\phi(x) = x \sin(1/x)$ for $x > 0$ and $\phi(0) = 0$, is strictly W-pseudoconvex at x^0, but not W-quasiconvex at x^0.

The following definition introduces jointly pseudoconvex (jointly strictly pseudoconvex) at a point functions and resembles Definition 4.3. Theorems 4.6 relates through scalarization a W-pseudoconvex (strictly W-pseudoconvex) at x^0 functions and jointly pseudoconvex (jointly strictly pseudoconvex) at x^0 functions. Let us underline, that when the scalar functions $\phi^j : X_0 \to \mathbb{R}$, $j \in J$, are pseudoconvex (strictly pseudoconvex) at x^0, they are also jointly pseudoconvex (jointly strictly pseudoconvex) at x^0 as it is seen from the following example. Let us underline however, that if X and T are finite dimensional (normed) spaces, W is polyhedral and $\phi : X_0 \to T$ is smooth, then if ϕ is W-pseudoconvex at x^0, it is also W-quasiconvex at x^0 (the smoothness is absent in Example 4.1).

Definition 4.6 We say that the scalar functions $\phi^j : X_0 \to \mathbb{R}$, $j \in J$, are jointly pseudoconvex (jointly strictly pseudoconvex) at $x^0 \in X_0$, if when for a point $x^1 \in X_0 \backslash \{x^0\}$ all the inequalities $\phi^j(x^1) < \phi^j(x^0)$ ($\phi^j(x^1) \leq \phi^j(x^0)$), $j \in J$, are satisfied, then $(\phi^i)'_-(x^0, x^1 - x^0) < 0$ holds for all $j \in J$.

Theorem 4.6 *If the function $\phi : X_0 \to T$ is W-pseudoconvex (strictly W-pseudoconvex) at $x^0 \in X_0$, then the functions $\langle \xi, \phi(x) \rangle$, $\xi \in W' \backslash \{0\}$, are jointly pseudoconvex (jointly strictly pseudoconvex) at x^0.*

Conversely, let T be locally convex space, and suppose that the functions $\langle \xi, \phi(x) \rangle$, $\xi \in W' \backslash \{0\}$, are jointly pseudoconvex (jointly strictly pseudoconvex) at x^0. Suppose also that the following property has place: when $\langle \xi, \phi \rangle'_-(x^0, u) < 0$ holds for all $\xi \in W' \backslash \{0\}$, then $\phi_-^{[1]}(x^0, u) \cap -\text{int}\, W \neq \emptyset$ (for instance, when $\phi_-^{[1]}(x^0, u)$ is a singleton for all $u \in X_0(x^0)$ this property is obviously satisfied). Then ϕ is pseudoconvex (strictly pseudoconvex) at x^0.

Proof Let the function $\phi : X_0 \to T$ be W-pseudoconvex (strictly W-pseudoconvex) at x^0. Take $x^1 \in X_0 \backslash \{x^0\}$ such that $\langle \xi, \phi(x^1)\rangle - \langle \xi, \phi(x^0)\rangle < 0$ ($\langle \xi, \phi(x^1)\rangle - \langle \xi, \phi(x^0)\rangle \le 0$) for all $\xi \in W' \backslash \{0\}$. This implies $\phi(x^1) - \phi(x^0) \in -\text{int}\, W$ ($\phi(x^1) - \phi(x^0) \in -W$), whence there exists a point $t^0 \in \phi_-^{[1]}(x^0, x^1 - x^0) \cap -\overline{\text{int}}\, \overline{W}$. Then for all $\xi \in W'$ it holds $\langle \xi, \phi\rangle'_-(x^0, x^1 - x^0) \le \langle \xi, t^0\rangle < 0$, whence the functions $\langle \xi, \phi(x)\rangle$, $\xi \in W' \backslash \{0\}$, are jointly pseudoconvex (strictly pseudoconvex) at x^0.

Conversely, let $x^1 \in X_0 \backslash \{x^0\}$ be such that $\phi(x^1) - \phi(x^0) \in -\text{int}\, W$ ($\phi(x^1) - \phi(x^0) \in -W$). Then $\langle \xi, \phi(x^1)\rangle - \langle \xi, \phi(x^0)\rangle < 0$ ($\langle \xi, \phi(x^1)\rangle - \langle \xi, \phi(x^0)\rangle \le 0$) for all $\xi \in W' \backslash \{0\}$. From the joint pseudoconvexity (joint strict pseudoconvexity) we have $\langle \xi, \phi\rangle'_-(x^0, x^1 - x^0) < 0$, $\xi \in W' \backslash \{0\}$. The hypotheses give $\phi_-^{[1]}(x^0, x^1 - x^0) \cap -\text{int}\, W \ne \emptyset$, which shows that ϕ is W-pseudoconvex (strictly W-pseudoconvex). □

Definition 4.5 generalizes the following notion of convexity.

Definition 4.7 The function $\phi : X_0 \to T$ is said W-convex (strictly W-convex) if for all $x^1, x^2 \in X_0$, $x^1 \ne x^2$, and all $t \in (0, 1)$ it holds $\phi((1-t)x^2 + tx^1) \in (1-t)\phi(x^2) + t\phi(x^1) - W$ ($\phi((1-t)x^2 + tx^1) \in (1-t)\phi(x^2) + t\phi(x^1) - \text{int}\, W$). When these properties hold for a fixed $x^2 = x^0$, then we say that ϕ is W-convex (strictly W-convex) at x^0.

The following theorem shows that the notion of W-pseudoconvexity (strict W-pseudoconvexity) can be considered as a generalization of the notion of W-convexity (strict W-convexity).

Theorem 4.7 *Let X and T be normed spaces, T finite dimensional, and $\phi : X_0 \to T$ Lipschitz near $x^0 \in X_0$. If the function $\phi : X_0 \to T$ is W-convex (strictly W-convex) at x^0, then ϕ is also W-pseudoconvex (strictly W-pseudoconvex) at x^0.*

Proof Let ϕ be W-convex at x^0. Choose $x^1 \in X_0 \backslash \{x^0\}$ such that $\phi(x^1) - \phi(x^0) \in -\text{int}\, W$. Then $\phi(x^1) - \phi(x^0) \in -w^0 - W$ for some $w^0 \in \text{int}\, W$. Let $t_k \to 0^+$. Since T is finite dimensional, according to Theorem 3.1 the extension $\overline{T}$ is compact. Passing eventually to a subsequence we may assume that $t^0 = \lim_k (1/t_k)\left(\phi(x^0 + t_k(x^1 - x^0)) - \phi(x^0)\right)$. Now $t^0 \in \phi_-^{[1]}(x^0, x^1 - x^0)$. We will show that $t^0 \in -\overline{\text{int}}\, \overline{W}$. In fact, the local Lipschitz condition gives that $t^0 \in T$, that is t^0 is finite. Hence, we have to show that $t^0 \in -\text{int}\, W$. Observe that when $0 < t_k < 1$ we have

$$\frac{1}{t_k}\left(\phi((1-t_k)x^0 + t_k x^1) - \phi(x^0)\right) \in \frac{1}{t_k}\left((1-t_k)\phi(x^0) + t_k\phi(x^1) - W - \phi(x^0)\right)$$
$$\subset \phi(x^1) - \phi(x^0) - W \subset -w^0 - W - W \subset -w^0 - W.$$

A passing to a limit gives $t^0 \in -w^0 - W \subset -\text{int}\, W$. Therefore ϕ is pseudoconvex at x^0.

Assume now that ϕ is strictly W-convex at x^0. Choose $x^1 \in X_0 \backslash \{x^0\}$ such that $\phi(x^1) - \phi(x^0) \in -W$. Put $\bar{x}^1 = (1/2)(x^0 + x^1)$. Then $\phi(\bar{x}^1) - \phi(x^0) \in \text{int}\, W$. Proceeding as above, we get that $\phi_-^{[1]}(x^0, \bar{x}^1 - x^0) \cap -\overline{\text{int}}\, \overline{W} \ne \emptyset$. With regard to $\phi_-^{[1]}(x^0, \bar{x}^1 - x^0) = (1/2)\, \phi_-^{[1]}(x^0, x^1 - x^0)$ we get that also $\phi_-^{[1]}(x^0, x^1 - x^0) \cap -\overline{\text{int}}\, \overline{W} \ne \emptyset$. Therefore ϕ is strictly W-pseudoconvex at x^0. □

In connection with the Lipschitz hypothesis in Theorem 4.7, let us recall that when ϕ is W-convex, X is finite dimensional, X_0 is open, and W is polyhedral, then ϕ is locally Lipschitz.

5 Necessary conditions

In this section we discuss necessary optimality conditions for problem (1) in terms of Dini directional derivatives.

In the next Theorem 5.1 (and only here) we give a different sense to the Dini derivative $(f,\ g)_-^{[1]}(x^0, u)$ than that from the accepted definition in Sect. 3 (where $(f,\ g)_-^{[1]}(x^0, u)$ means the Dini derivative of the function $\phi = (f,\ g)$ and whose values are in $\overline{Y \times Z}$). We put $(f,\ g)_-^{[1]}(x^0, u)$ to be the set of all $(y, z) \in \overline{Y} \times \overline{Z}$ such that for some sequence of reals $t_k \to 0^+$ it holds

$$y = \lim_k \frac{1}{t_k}\left(f(x^0 + t_k u) - f(x^0)\right), \quad z = \lim_k \frac{1}{t_k}\left(g(x^0 + t_k u) - g(x^0)\right).$$

A particular case of Theorem 5.1 dealing with locally Lipschitz functions f and g appears in [13]. Let us underline that the differential quotient of a locally Lipschitz function is bounded by the Lipschitz constant near the considered point, which makes redundant the use of infinite elements in the definition of the Dini derivative. Here Theorem 5.1, which represents a result similar to that of [13], is used rather as an issue point to explain the evolution when we pass to problems with quasiconvex constraints, which consists in the possibility to replace the derivative $(f,\ g)_-^{[1]}(x^0, u)$ with $f_-^{[1]}(x^0, u) \times g_-^{[1]}(x^0, u)$.

Theorem 5.1 *Let Z be a normed space and let K' have a compact base Γ. Consider problem (1). Let x^0 be a radial w-minimizer of (1) and let g be radially continuous at x^0. Then for each $u \in X_0(x^0)$ it holds*

$$(f,\ g)_-^{[1]}(x^0, u) \cap (-\overline{\text{int}}\,(\overline{C} \times \overline{K[-g(x^0)]})) = \emptyset. \tag{2}$$

Proof Suppose on the contrary, that for some $u^0 \in X_0(x^0)$ there exists $(\bar{y}^0, \bar{z}^0) \in (f,\ g)_-^{[1]}(x^0, u^0)$ such that $\bar{y}^0 \in -\overline{\text{int}}\ \overline{C}$, $\bar{z}^0 \in -\overline{\text{int}}\ \overline{K[-z^0]}$. Let $\bar{y}^0 = \lim_k (1/t_k)(y^k - y^0)$ and $\bar{z}^0 = \lim_k (1/t_k)(z^k - z^0)$ for some sequence $t_k \to 0^+$, where $y^k = f(x^0 + t_k u^0)$, $y^0 = f(x^0)$, $z^k = g(x^0 + t_k u^0)$, $z^0 = g(x^0)$.

Now we proof that the points $x^0 + t_k u^0$ are feasible for all sufficiently large k. Let $\bar{\eta} \in \Gamma$. We show that there exists a positive integer $k(\bar{\eta})$ and a neighbourhood $V(\bar{\eta})$ of $\bar{\eta}$, such that $\langle \eta, z^k \rangle < 0$ for $k > k(\bar{\eta})$ and $\eta \in V(\bar{\eta})$. For this purpose we consider the cases:

1^0. $\bar{\eta} \in K'[-z^0]$. Since $\bar{z}^0 \in -\overline{\text{int}}\ \overline{K[-z^0]}$, we have $(1/t_k)(z^k - z^0) \in -\text{int}\, K[-z^0]$ for all sufficiently large k, whence $z^k - z^0 \in -\text{int}\, K[-z^0]$. Therefore $z^k - z^0 + \varepsilon B \subset -\text{int}\, K[-z^0]$ for some $\varepsilon > 0$ and all sufficiently large k. Here B is the unit ball in Z. This gives $\langle \bar{\eta}, z^k \rangle = \langle \bar{\eta}, z^k - z^0 \rangle \le -\varepsilon \|\bar{\eta}\|$. Let $\|\eta - \bar{\eta}\| < \varepsilon\, \|\bar{\eta}\| / \sup_k \|z^k\|$ (pay attention that the sequence $\|z^k\|$ is bounded because from the radial continuity of g at x^0 we have $z^k \to z^0$). Now

$$\langle \eta, z^k \rangle = \langle \eta - \bar{\eta}, z^k \rangle + \langle \bar{\eta}, z^k \rangle \le \|\eta - \bar{\eta}\|\, \|z^k\| - \varepsilon \|\bar{\eta}\| < 0.$$

2^0. $\bar{\eta} \in K' \backslash K'[-z^0]$. Now $\langle \bar{\eta}, z^0 \rangle < -\varepsilon\, \|\bar{\eta}\|$ for some $\varepsilon > 0$, whence $\langle \bar{\eta}, z^k \rangle < -\varepsilon\, \|\bar{\eta}\|$ for all sufficiently large k. Let $\|\eta - \bar{\eta}\| < \varepsilon\, \|\bar{\eta}\| / \sup_k \|z^k\|$. As in case 1^0 we get $\langle \eta, z^k \rangle < 0$.

The compactness of Γ gives $\Gamma \subset V(\bar{\eta}^1) \cup \cdots \cup V(\bar{\eta}^s)$ for some $\bar{\eta}^1, \ldots, \bar{\eta}^s \in \Gamma$. Let $k_0 = \max(k(\bar{\eta}^1), \ldots, k(\bar{\eta}^s))$. Take $k > k_0$. Then $\langle \eta, z^k \rangle < 0$ for all $\eta \in \Gamma$, and hence for all $\eta \in K' \backslash \{0\}$. Therefore $z^k \in -\text{int}\, K \subset -K$, in other words, the points $x^0 + t_k u^0$ are feasible.

According to the made assumption $\bar{y}^0 = \lim_k (1/t_k)(y^k - y^0) \in -\overline{\text{int}}\ \overline{C}$. Therefore $y^k - y^0 \in -\text{int}\, C$ for all sufficiently large k, a contradiction to the hypothesis that (x^0, y^0) is a radial w-minimizer of (1). □

If int $K \neq \emptyset$ then K' admits a bounded and hence weak* compact base Γ [16]. This observation raises the question whether in Theorem 5.1 the hypothesis that "K' possesses a compact base" can be replaced by "K' possesses a weak* compact base". The following example gives a negative answer.

Example 5.1 Consider problem (1) with $X = \mathbb{R}$, $X_0 = \mathbb{R}_+$, $Y = \mathbb{R}$, $C = \mathbb{R}_+$, $Z = c$ being the Banach space of the bounded sequences $z = (z_1, z_2 \dots)$ supplied with the norm $\|z\| = \sup_n |z_n|$, $K = c_+ = \{z \in Z \mid z_i \geq 0 \ (i = 1, 2, \dots)\}$ (with int $K \neq \emptyset$), $f : X_0 \to \mathbb{R}$ defined by $f(t) = -t$, $g : X_0 \to Z$ defined by $g(t) = -w^0 + t\, e$ where $w^0 \in Z$ has positive components $w_i^0 > 0$ $(i = 1, 2, \dots)$ such that $\lim_i w_i^0 = 0$ and $e = (1, 1, \dots)$ has all components 1, and $x^0 = 0$. We have

$$Z^* = \ell^1 := \{\eta = (\eta_1, \eta_2, \dots) \mid \|\eta\|_{\ell^1} = \sum_{i=1}^{\infty} |\eta_i| < \infty\},$$

$K' = \ell^1_+ = \{\eta \in Z^* \mid \eta_i \geq 0\, (i = 1, 2, \dots)\}$, and $X_0(x^0) = \mathbb{R}_+$. The point x^0 is a radial w-minimizer of (1), since it is the only feasible point. Obviously $f'(x^0, 1) = -1 \in -\text{int}\, C$, $g'(x^0, 1) = e \in Z = -\text{int}\, K[w^0]$, whence condition (2) does not hold.

The inclusion

$$(f, g)^{[1]}_-(x^0, u) \subset f^{[1]}_-(x^0, u) \times g^{[1]}_-(x^0, u) \tag{3}$$

has place (where the derivative $(f, g)^{[1]}_-(x^0, u)$ is understood as in Theorem 5.1), but usually for nonsmooth functions these sets are different. So, in general condition (2) in the thesis of Theorem 5.1 cannot be replaced by

$$f^{[1]}_-(x^0, u) \times g^{[1]}_-(x^0, u) \cap \left(-\overline{\text{int}}\,(\overline{C} \times \overline{K[-g(x^0)]}\,)\right) = \emptyset. \tag{4}$$

This is illustrated by the following example.

Example 5.2 Consider problem (1) with $X = \mathbb{R}$, $X_0 = \mathbb{R}_+$, $Y = \mathbb{R}$, $C = \mathbb{R}_+$, $Z = \mathbb{R}$, $K = \mathbb{R}_+$, $f : X_0 \to Y$ defined by

$$f(x) = \begin{cases} x \sin(1/x)\,, & x > 0\,, \\ 0\,, & x = 0\,, \end{cases}$$

and $g : X_0 \to Z$ defined by $g(x) = -f(x)$. Let $x^0 = 0$. Then $f(x^0) = 0$ and x^0 is a w-minimizer of problem (1). We have $g(x^0) = 0$ and $K[-g(x^0)] = K$. Condition (2) is satisfied, since $(\bar{y}^0, \bar{z}^0) \in (f, g)^{[1]}_-(x^0, u)$, $u \in X_0(x^0)$, implies $\bar{z}^0 = -\bar{y}^0$, whence

$$(\bar{y}^0, \bar{z}^0) = (\bar{y}^0, -\bar{y}^0) \notin [-\infty, 0) \times [-\infty, 0) = -\overline{\text{int}}\left(\overline{C} \times \overline{K[-g(x^0)]}\right).$$

At the same time condition (4) does not hold, since $f^{[1]}_-(x^0, 1) = g^{[1]}_-(x^0, 1) = [-1, 1]$.

When $f^{[1]}_-(x^0, u)$ or $g^{[1]}_-(x^0, u)$ is a singleton, then inclusion (3) turns into an equality. Consequently, if this holds for all $u \in X_0(x^0)$ Theorem 5.1 is true with condition (4) instead of (2). Example 5.2 shows that in general this does not hold. In Sects. 7 and 8 the satisfaction of condition (4) allows in the considered there particular problems to substitute the Dini set valued derivative with the more simple single valued lower Dini derivative. Hence the implementation of (4) instead of (2) is more relevant with regard of the aim of the investigation. Actually, dealing with problems with quasiconvex constraints, we get optimality conditions involving (4), and this is the reason why we occupy here with problems with such constraints. The next Theorem 5.3 illustrates this idea.

Theorem 5.2 *Consider problem (1). Let x^0 be a radial w-minimizer of (1) and suppose that for all $u \in X_0(x^0)$ the following constraint qualification of Kuhn-Tucker type holds:*

$$\mathbb{Q}(x^0, u): \quad \begin{cases} \textit{If } g(x^0 + t_0 u) \in -\operatorname{int} K[-g(x^0)] \textit{ for some } t_0 > 0 \\ \textit{then there exists } \bar{t} > 0 \textit{ such that } g(x^0 + \bar{t}u) \in -K\,. \end{cases}$$

Let the function g be K-quasiconvex. Then for each $u \in X_0(x^0)$ condition (4) holds.

Proof Suppose on the contrary, that for some $u^0 \in X_0(x^0)$ there exist $\bar{y}^0 \in f_-^{[1]}(x^0, u^0)$ and $\bar{z}^0 \in g_-^{[1]}(x^0, u^0)$ such that $\bar{y}^0 \in -\overline{\operatorname{int}}\ \overline{C}$ and $\bar{z}^0 \in -\overline{\operatorname{int}}\ \overline{K[-z^0]}$. Let $\bar{y}^0 = \lim_k (1/s_k)(y^k - y^0)$ and $\bar{z}^0 = \lim_k (1/t_k)(z^k - z^0)$ for some sequences $s_k \to 0^+$ and $t_k \to 0^+$, where $y^k = f(x^0+s_k u^0)$, $y^0 = f(x^0)$, $z^k = g(x^0+t_k u^0)$, $z^0 = g(x^0)$. Then there exists a positive integer k_0 such that $(1/t_{k_0})(z^{k_0}-z^0) \in -\operatorname{int} K[-g(x^0)]$ (the bars in $-\overline{\operatorname{int}}\ \overline{K[-g(x^0)]}$ can be dropped because $(1/t_{k_0})(z^{k_0} - z^0)$ is finite), whence $g(x^0 + t_{k_0}u^0) = z^{k_0} \in z^0 - \operatorname{int} K[-g(x^0)] \subset -K - \operatorname{int} K[-g(x^0)] \subset -K[-g(x^0)] - \operatorname{int} K[-g(x^0)] \subset -\operatorname{int} K[-g(x^0)]$. The constraint qualification $\mathbb{Q}(x^0, u^0)$ gives that there exists $\bar{t} > 0$ such that $g(x^0 + \bar{t}u^0) \in -K$. Since also $g(x^0) = z^0 \in -K$ and g is K-quasiconvex, we have $g(x^0 + tu^0) \in -K$ for all $t \in [0, \bar{t}]$. Choose the positive integer $\bar{k}$ such that $s_k < \bar{t}$ for all $k \geq \bar{k}$. The points $x^0 + s_k u^0$, $k \geq \bar{k}$, are feasible and $(1/s_k)(y^k - y^0) \in -\operatorname{int} C$ (the bars in $-\overline{\operatorname{int}}\ \overline{C}$ can be dropped because $(1/s_k)(y^k - y^0)$ are finite), whence $f(x^0 + s_k u^0) \in f(x^0) - \operatorname{int} C$. This contradicts the hypothesis that x^0 is a radial w-minimizer. □

Remark 5.1 From the hypothesis that g is K-quasiconvex, that is that the sets $g^{-1}(z - K) = \{x \in X_0 \mid g(x) \in z - K\}$, for all $z \in Z$, are convex, we have used the convexity only when $z = 0$, that is the convexity of the set $g^{-1}(-K)$. Even more, we have used only that $g^{-1}(0)$ is star-shaped with respect to x^0. The latter means that for all $x \in g^{-1}(-K)$ the segment $[x^0, x]$ is contained in $g^{-1}(-K)$. Therefore, in Theorem 5.2 the hypothesis "g is K-quasiconvex" can be replaced by the weaker one "$g^{-1}(-K)$ is star-shaped with respect to x^0".

The next theorem establishes necessary optimality conditions with no constraint qualifications involved. It uses the less restrictive quasiconvexity hypothesis that g is $K[-g(x^0)]$-quasiconvex instead of K-quasiconvex. The price we pay is the more restrictive hypothesis that the cone K' is polyhedral. It remains an open question whether this theorem remains true when the hypothesis "K' is polyhedral" is substituted by the weaker one "K' has a compact base". In the case when Y is a normed space if $f_-^{[1]}(x^0, u)$ or $g_-^{[1]}(x^0, u)$ is a singleton for all $u \in X_0(x^0)$, and "g is radially continuous at x^0", the positive answer is given by Theorem 5.1, since now $(f, g)_-^{[1]}(x^0, u) = f_-^{[1]}(x^0, u) \times g_-^{[1]}(x^0, u)$. Let us still underline that for us the case of K' polyhedral is of special interest, since the positive orthant cones used in the considered particular problems in Sects. 7 and 8 are of this type.

Theorem 5.3 *Let Z be a locally convex space and let the cone K' be polyhedral. Consider problem (1). Let x^0 be a radial w-minimizer of (1). Let g be $K[-g(x^0)]$-quasiconvex at x^0, and let the functions $\langle \eta, g\rangle$ be radially continuous at x^0 for all $\eta \in$* extd *K' such that $\langle \eta, g(x^0)\rangle \neq 0$. Then for each $u \in X_0(x^0)$ condition (4) is satisfied.*

Proof Suppose on the contrary, that for some $u^0 \in X$ there exists $\bar{y}^0 \in f_-^{[1]}(x^0, u^0)$, $\bar{z}^0 \in g_-^{[1]}(x^0, u^0)$ such that $\bar{y}^0 \in -\overline{\operatorname{int}}\ \overline{C}$, $\bar{z}^0 \in -\overline{\operatorname{int}}\ \overline{K[-z^0]}$. Let $\bar{y}^0 = \lim_k (1/s_k)(y^k - y^0)$ and $\bar{z}^0 = \lim_k (1/t_k)(z^k - z^0)$ for some sequences $s_k \to 0^+$ and $t_k \to 0^+$, where $y^k = f(x^0 + s_k u^0)$, $y^0 = f(x^0)$, $z^k = g(x^0 + t_k u^0)$, $z^0 = g(x^0)$. Since the cone K' is polyhedral, it possesses a compact base of the type $\Gamma = \operatorname{co}\{\eta^1, \ldots, \eta^q\}$. We claim that for each

η^i $(i = 1, \dots q)$ there exists k_i such that $\langle \eta^i, g(x^0 + tu^0) \rangle \le 0$ for $0 \le t \le t_{k_i}$. When $\langle \eta^i, g(x^0) \rangle = 0$ it is enough to take k_i such that $(1/t_{k_i})(z^{k_i} - z^0) \in -\text{int}\, K[-g(x^0)]$. Then $\langle \eta^i, z^{k_i} \rangle \le 0$ and the $K[-g(x^0)]$-quasiconvexity at x^0 of g gives

$$\langle \eta^i, g(x^0 + tu^0) \rangle \le \max \Big(\langle \eta^i, g(x^0) \rangle,\ \langle \eta^i, g(x^0 + t^k u^0) \rangle \Big) \le 0\,.$$

When $\langle \eta^i, g(x^0) \rangle < 0$ our claim follows from the radial continuity of $\langle \eta^i, g \rangle$. Put $k^0 = \max\{k_1, \dots, k_q\}$. We have proved that all the points $x^0 + tu^0$, $0 \le t \le t_{k_0}$, are feasible. Therefore the points $x^0 + s_k u^0$ are feasible for all sufficiently large k. Now we get $y^k \in y^0 - \text{int}\, C$, a contradiction to the hypothesis that x^0 is a radial w-minimizer of (1). □

Remark 5.2 Using the notation from the proof, introduce the set of the active indexes $J(x^0) = \{j \mid \langle \eta^j, g(x^0) \rangle = 0\}$. The $K[-g(x^0)]$-quasiconvexity at x^0 of g according to Theorems 4.1 and 4.5 is equivalent to the joint quasiconvexity at x^0 of the functions $\langle \eta^j, g(x) \rangle$, $j \in J(x^0)$. The radial continuity at x^0 is assumed only for the functions $\langle \eta^j, g(x) \rangle$, $j \notin J(x^0)$.

Condition (4) can be referred as optimality condition in primal form. The next theorem establishes that in important cases it is equivalent to condition (5), which can be referred as optimality condition in dual form. Similarly, replacing in the first row of (5) $f_-^{[1]}(x^0, u) \times g_-^{[1]}(x^0, u)$ with $(f, g)_-^{[1]}(x^0, u)$ we get the equivalent dual condition of (the primal) condition (2). The second row in (5) can be written in the form $\exists\, (\xi^0, \eta^0) \in C' \times K'[-g(x^0)]$. We prefer a record exposing the slackness condition $\langle \eta^0, g(x^0) \rangle = 0$. The sum $\langle \xi^0, \bar{y} \rangle + \langle \eta^0, \bar{z} \rangle$ has always sense, for neither of its addends takes value $-\infty$.

Theorem 5.4 *When Y and Z are locally convex spaces, and C and K have nonempty interiors,* (*or when Y and Z are finite dimensional*) *condition* (4) *is equivalent to:*

$$\begin{array}{c} \forall\, (\bar{y}, \bar{z}) \in f_-^{[1]}(x^0, u) \times g_-^{[1]}(x^0, u) : \\ \exists\, (\xi^0, \eta^0) \in C' \times K' : \quad \langle \eta^0, g(x^0) \rangle = 0, \\ (\xi^0, \eta^0) \ne (0,\ 0), \quad \langle \xi^0, \bar{y} \rangle \ne -\infty, \quad \langle \eta^0, \bar{z} \rangle \ne -\infty, \\ \textit{and} \quad \langle \xi^0, \bar{y} \rangle + \langle \eta^0, \bar{z} \rangle \ge 0\,. \end{array} \tag{5}$$

Proof Let $(\bar{y}, \bar{z}) \in f_-^{[1]}(x^0, u) \times f_-^{[1]}(x^0, u)$. Take the couple $(y^0, z^0) \in Y \times Z$, such that $y^0 = \bar{y}$ when $\bar{y} \in Y$ or $(y^0)_\infty = \bar{y}$ when $\bar{y} \in Y_\infty$, similarly $z^0 = \bar{z}$ when $\bar{z} \in Z$ or $(z^0)_\infty = \bar{z}$ when $\bar{z} \in Z_\infty$. Obviously, condition (4) is equivalent to $(y^0, z^0) \notin -(\text{int}\, C \times K[-g(x^0)])$ for any possible choice of $(\bar{y}, \bar{z})$. Applying the Separation Theorem, we see that this is equivalent to the existence of $(\xi^0, \eta^0) \in Y^* \times Z^*$, $(\xi^0, \eta^0) \ne (0,\ 0)$, such that $\langle \xi^0, y^0 \rangle + \langle \eta^0, z^0 \rangle \ge 0$ and $\langle \xi^0, y \rangle + \langle \eta^0, z \rangle \le 0$ when $(y, z) \in -(C \times K[-g(x^0)])$. Moreover, the couple (ξ^0, η^0) can be chosen so that $\langle \xi^0, y^0 \rangle \ge 0$ and $\langle \eta^0, z^0 \rangle \ge 0$ (when $y^0 \notin -\text{int}\, C$ we may put $\eta^0 = 0$, or when $z^0 \notin -\text{int}\, K[-g(x^0)]$ we may put $\xi^0 = 0$). The inequalities $\langle \xi^0, y \rangle + \langle \eta^0, z \rangle \le 0$, $(y, z) \in -(C \times K[-g(x^0)])$, with regard that C and $K[-g(z^0)]$ are cones, give $\langle \xi^0, y \rangle \ge 0$, $\forall\, y \in C$, and $\langle \eta^0, z \rangle \ge 0$, $\forall\, z \in K[-g(x^0)]$, that is $\xi^0 \in C'$ and $\eta^0 \in K'[-g(x^0)]$. Now $\langle \xi^0, y^0 \rangle \ge 0$ gives $\langle \xi^0, \bar{y} \rangle \ge 0$, and $\langle \eta^0, z^0 \rangle \ge 0$ gives $\langle \eta^0, \bar{z} \rangle \ge 0$. Hence $\langle \xi^0, \bar{y} \rangle \ne -\infty$, $\langle \eta^0, \bar{z} \rangle \ne -\infty$, and $\langle \xi^0, \bar{y} \rangle + \langle \eta^0, \bar{z} \rangle \ge 0$. □

6 Sufficient conditions and global w-minimizers

In this section we are interested to distinguish classes of functions, for which condition (4) is sufficient for the optimality of the reference point x^0. The pseudoconvexity plays an important role in these considerations. Under pseudoconvexity assumptions the optima turn to be global w-minimizers.

Theorem 6.1 *Let Z be a locally convex space. Consider problem (1). Let g be strictly $K[-g(x^0)]$-pseudoconvex at x^0 and f be C-pseudoconvex at x^0. Suppose that for each $u \in X_0(x^0)$ condition (4) is satisfied. Then x^0 is a global w-minimizer of problem (1). If in addition f is strictly C-pseudoconvex at x^0, then x^0 is a strict global w-minimizer.*

Proof Assume on the contrary, that x^0 is not a global w-minimizer. Then there exists a feasible point $x^1 \in X_0$ such that $f(x^1) - f(x^0) \in -\text{int}\, C$. Since f is C-pseudoconvex at x^0, it holds $f_-^{[1]}(x^0, u) \cap (-\overline{\text{int}}\ \overline{C}) \neq \emptyset$ with $u = x^1 - x^0$. Therefore condition (4) gives that $g_-^{[1]}(x^0, u) \cap -\overline{\text{int}}\ \overline{K[-g(x^0)]} = \emptyset$. On the other hand $g(x^1) - g(x^0) \in -K[-g(x^0)]$. Indeed, for all $\eta \in K'[-g(x^0)]$ we have

$$\langle \eta, g(x^1) - g(x^0) \rangle = \langle \eta, g(x^1) \rangle \leq 0$$

(here we use that $K[-g(x^0)]$ coincides with its second positive polar cone, a consequence of Z locally convex space). Since g is strictly $K[-g(x^0)]$-pseudoconvex at x^0, we have $g_-^{[1]}(x^0, u) \cap -\overline{\text{int}}\ \overline{K[-g(x^0)]} \neq \emptyset$, a contradiction. When f is strictly C-pseudoconvex, the global minimizer x^0 is strict. Indeed, on the contrary we would have $f(x^1) - f(x^0) \in -C$ for some feasible point $x^1 \in X_0 \backslash \{x^0\}$. Put $u = x^1 - x^0$. The strict C-pseudoconvexity of f gives $f_-^{[1]}(x^0, u) \cap (-\overline{\text{int}}\ \overline{C}) \neq \emptyset$, and the strict $K[-g(x^0)]$-pseudoconvexity of g gives $g_-^{[1]}(x^0, u) \cap -\overline{\text{int}}\ \overline{K[-g(x^0)]} \neq \emptyset$, a contradiction. □

The following Theorem 6.2 is a direct consequence of Theorem 6.1 because of the relation between convexity and pseudoconvexity given in Theorem 4.7. Similar statement one finds in [8]. It can be considered as a variant of the classical result claiming that any Kuhn-Tucker point in a convex programming problem is a global minimizer. In this sense we recognize that the obtained here sufficient conditions generalize to vector optimization classical results from convex programming. We concentrate on quasiconvex constraints following some direction in mathematical programming. Quasiconvex programming, initiating with the study of scalar smooth quasiconvex programming problems, has a long history. A parallel between convex and quasiconvex programming one finds in Luenberger [18]. It is worth mentioning some similarity of Theorem 6.1 with other classical results concerning scalar nonsmooth problems, see e. g. Arrow, Enthoven [2,Theorem 3], Bair [3,Proposition 3], Bector et al. [4,Theorem 3.1], Giorgi [15,Theorem 1.4].

Theorem 6.2 *Let X, Y and Z be normed spaces, Y and Z finite dimensional, and f and g Lipschitz near x^0. Let g be strictly $K[-g(x^0)]$-convex at x^0 and f be C-convex at x^0. Suppose that for each $u \in X_0(x^0)$ condition (4) is satisfied. Then x^0 is a global w-minimizer of problem (1). If in addition f is strictly C-convex at x^0, then x^0 is a strict global w-minimizer.*

Theorem 6.1 in comparison with the respective necessary condition ("respective" in the sense that they both concern the cone $K[-g(x^0)]$) from Theorem 5.3 does not confine to polyhedral cones K and does not use for g radial continuity conditions at x^0. The next theorem deals with K-pseudoconvexity, so it is respective to Theorem 5.2. Observe that now the hypotheses are similar to these of Theorem 5.3.

Theorem 6.3 *Let Z be a locally convex space and let the cone K' be polyhedral. Consider problem (1). Let g be strictly K-pseudoconvex at x^0 and f be C-pseudoconvex at x^0. Let also g be $K[-g(x^0)]$-quasiconvex at x^0 and $\langle \eta, g \rangle$ be radially continuous at x^0 for any $\eta \in \text{extd}\, K$ such that $\langle \eta, g(x^0) \rangle \neq 0$. Suppose that for each $u \in X_0(x^0)$ condition (4) is satisfied. Then x^0 is a global w-minimizer of problem (1). If in addition f is strictly C-pseudoconvex at x^0, then x^0 is a strict global w-minimizer.*

Proof Assume on the contrary, that x^0 is not a global w-minimizer. Then there exists a feasible point $x^1 \in X_0$ such that $f(x^1) - f(x^0) \in -\text{int}\, C$. Since f is C-pseudoconvex at x^0, it holds $f_-^{[1]}(x^0, u) \cap (-\overline{\text{int}}\ \overline{C}) \neq \emptyset$ with $u = x^1 - x^0$. Therefore condition (4) gives that $g_-^{[1]}(x^0, u) \cap -\overline{\text{int}}\ \overline{K[-g(x^0)]} = \emptyset$. On the other hand $g(x^0 + tu) - g(x^0) \in -K$ for some $t > 0$. To show this let $\Gamma = \text{co}\,\{\eta^1, \dots, \eta^n\}$ be a base of K. When $\eta \in K'[-g(x^0)]$ from $g(x^1) \in -K$ we have $\langle \eta, g(x^1) - g(x^0)\rangle = \langle \eta, g(x^1)\rangle \leq 0$. The $K[-g(x^0)]$-quasiconvexity at x^0 of g gives now $\langle \eta, g(x^0 + tu)\rangle \leq 0$ for all $t \in [0,\ 1]$. When $\eta \in K' \backslash K'[-g(x^0)]$ the radial continuity at x^0 of $\langle \eta, g\rangle$ with regard to $\langle \eta, g(x^0)\rangle < 0$ gives $\langle \eta, g(x^0 + tu)\rangle < 0$ for all sufficiently small t. Thus, we can find t such that $\langle \eta^i, g(x^0 + tu)\rangle \leq 0$ for all $i = 1, \dots, n$. Since g is strictly $K[-g(x^0)]$-pseudoconvex at x^0, we have $g_-^{[1]}(x^0, u) \cap -\overline{\text{int}}\ \overline{K[-g(x^0)]} \neq \emptyset$, a contradiction. When f is strictly C-pseudoconvex, like in Theorem 6.1 we see that the global minimizer x^0 is strict. □

7 The positive orthant as ordering cone

In this section we reformulate the previous results for problem (1) with finite dimensional image space $Z = \mathbb{R}^p$ (with Euclidean norm), and ordering cone being the positive orthant $K = \mathbb{R}^p_+$ (then $K' = K$ has a base the convex hull of the unit vectors ξ^j along the axes). We write $g = (g_1, \dots, g_p)$, agreeing that in this and similar notations the lower indexes stand for the coordinates. We agree also that $0 \cdot \{\pm\infty\} = 0$. For $x^0 \in X_0$ the set of the active indexes for problem (1) is defined by $J(x^0) = \{j \mid g_j(x^0) = 0\}$. The main feature of this section is that we replace the set valued derivatives of g used in the previous sections with the single valued lower Dini derivatives.

Theorem 7.1 *Consider problem (1) with $Z = \mathbb{R}^p$ and $K = \mathbb{R}^p_+$ and let x^0 be a radial w-minimizer. Let the functions g_j, $j = 1, \dots, p$, be radially continuous at x^0 when $j \notin J(x^0)$ and quasiconvex at x^0 when $j \in J(x^0)$ (or less restrictive, $g_j(x)$, $j \in J(x^0)$, jointly quasiconvex at x^0). Then for each $u \in X_0(x^0)$ the following condition is satisfied:*

$$f_-^{[1]}(x^0, u) \times \{g'_-(x^0, u)\} \cap \left(-\overline{\text{int}} \left(\overline{C} \times \prod_{j=1}^{p} \overline{\mathbb{R}_+[-g_j(x^0)]} \right) \right) = \emptyset . \tag{6}$$

Proof We just check that the hypotheses of Theorem 5.3 are satisfied. Theorem 4.5 gives that g is $K[-g(x^0)]$-quasiconvex at x^0. The assumed radial continuity at x^0 of g_j, $j \notin J(x^0)$, coincides with the radial continuity at x^0 from the hypothesis of Theorem 5.3. Assume that (6) is not true. Then there are sequences $t_{jk} \to 0^+$ $(j = 1, \dots, p)$ and $s_k \to 0^+$, such that $\bar{y}^0 = \lim_k (1/s_k)(f(x^0 + s_k u) - f(x^0)) \in -\overline{\text{int}}\ \overline{C}$ and $\bar{z}_j^0 = \lim_k (1/t_{jk})(g_j(x^0 + t_{jk}u) - g_j(x^0)) \in -\overline{\text{int}}\ \overline{\mathbb{R}_+[-g_j(x^0)]}$ $(j = 1, \dots, p)$. Take a sequence $t_k \to 0^+$, such that $t_k < \min(t_{1k}, \dots, t_{pk})$. Passing to a subsequence, due to Z finite dimensional, we may assume that $\lim_k (1/t_k)(g(x^0 + t_k u) - g(x^0)) = \hat{z}^0$. The quasiconvexity assumption for g_j, $j \in J(x^0)$, gives $\hat{z}_j^0 \leq \bar{z}_j^0$ for $j \in J(x^0)$. This implies that $(\bar{y}^0, \hat{z}^0)$ belongs to the left-hand side set in (4), a contradiction with the thesis of Theorem 5.3. □

Remark 7.1 The primal form condition (6) admits an equivalent dual form representation (the equivalency is proved like in Theorem 5.4):

$$\begin{gathered}\forall\, \bar{y} \in f_{-}^{[1]}(x^0, u) : \\ \exists\, (\xi^0, \eta^0) \in C' \times \mathbb{R}_+^p : \quad \eta_j^0\, g_j(x^0) = 0 \ \ (j = 1, \dots, p), \\ (\xi^0, \eta^0) \neq (0,\, 0), \quad \langle \xi^0, \bar{y} \rangle \neq -\infty, \\ \eta_j^0 = 0 \ \text{ if } \ {g_j}'_{-}(x^0, u) = -\infty \ \ (j = 1, \dots, p), \\ \text{and} \quad \langle \xi^0, \bar{y} \rangle + \textstyle\sum_{j=1}^{p} \eta_j^0\, {g_j}'_{-}(x^0, u) \geq 0\,. \end{gathered}$$

Turn attention that in the sum in the last row we have $\eta_j^0\, {g_j}'_{-}(x^0, u) = 0$ when ${g_j}'_{-}(x^0, u) = -\infty$. So, the sum does not contain addends $-\infty$, hence it has always sense.

The following theorem gives sufficient conditions and is a straightforward corollary of Theorems 6.1 and 4.6.

Theorem 7.2 *Consider problem (1) with $Z = \mathbb{R}^p$ and $K = \mathbb{R}_+^p$. Let the functions g_j, $j \in J(x^0)$, be jointly strictly pseudoconvex at x^0, and f be C-pseudoconvex (strictly C-pseudoconvex) at x^0. Suppose also that when $(g_j)'_{-}(x^0, u) < 0$ holds for all $j \in J(x^0)$, then there is a sequence $t_k \to 0^+$ such that the following limits exist and satisfy the given inequalities $\lim_k \frac{1}{t_k} \big(g_j(x^0 + t_k u) - g_j(x^0)\big) < 0$. Suppose that for each $u \in X_0(x^0)$ condition (6) is satisfied. Then x^0 is a global w-minimizer (strict global w-minimizer).*

Remark 7.2 When $Y = \mathbb{R}^m$ and $C = \mathbb{R}_+^m$ also the set valued derivative $f_{-}^{[1]}(x^0, u)$ used in Theorems 7.1 and 7.2 can be replaced by the single valued lower Dini derivative. Actually the primal condition (6) becomes now

$$(f'_{-}(x^0, u),\, g'_{-}(x^0, u)) \notin -\overline{\text{int}} \left(\overline{\mathbb{R}}_+^m \times \prod_{j=1}^{p} \overline{\mathbb{R}_+[-g_j(x^0)]} \right)$$

and the equivalent dual condition

$$\begin{gathered} \exists\, (\xi^0, \eta^0) \in \mathbb{R}_+^m \times \mathbb{R}_+^p : \quad \eta_j^0\, g_j(x^0) = 0 \ \ (j = 1, \dots, p), \\ (\xi^0, \eta^0) \neq (0,\, 0), \quad \xi_i^0 = 0 \ \text{ if } \ {f_i}'_{-}(x^0, u) = -\infty \ \ (i = 1, \dots, m), \\ \eta_j^0 = 0 \ \text{ if } \ {g_j}'_{-}(x^0, u) = -\infty \ \ (j = 1, \dots, p), \\ \text{and} \quad \textstyle\sum_{i=1}^{m} \xi_i^0\, {f_i}'_{-}(x^0, u) + \sum_{j=1}^{p} \eta_j^0\, {g_j}'_{-}(x^0, u) \geq 0\,. \end{gathered}$$

8 The scalar problem

In this section we consider the scalar constrained optimization problem

$$\min f(x), \quad g_j(x) \leq 0 \quad (j = 1, \dots, p)\,, \tag{7}$$

where $f : X_0 \to \mathbb{R}$, $g_j : X_0 \to \mathbb{R}$ $(j = 1, \dots, p)$. Putting $g = (g_1, \dots, g_p)$ and agreeing to write $g(x) \leq 0$ when the coordinate functions satisfy the same inequality, we can write (7) in the form

$$\min f(x), \quad g(x) \leq 0\,. \tag{8}$$

Problem (7) is in fact a particular vector problem (1) with $Y = \mathbb{R}$, $C = \mathbb{R}_+$, $Z = \mathbb{R}^p$ and $K = \mathbb{R}_+^p$. The ordering cones are the positive orthants. Therefore to this problem we can apply the results from the previous section. Here we do it explicitly in the following Theorems 8.1 and 8.2. We repeat in some sense the results from the previous section, because of the importance of the scalar problems in optimization theory. Another aim of this section is to give some examples supporting the theory. Establishing necessary and sufficient conditions, we imposed some hypotheses on the involved functions. One may ask in how far these hypotheses are essential. The answer to some of theses questions is given by examples of scalar problems. Here we insert several such examples.

We formulate our results applying the following condition in primal form:

$$(f'_-(x^0,u),\, g'_-(x^0,u)) \notin -\overline{\mathrm{int}}\left(\overline{\mathbb{R}}_+ \times \prod_{j=1}^{p} \overline{\mathbb{R}_+[-g_j(x^0)]}\right). \tag{9}$$

Condition (9) admits replacement with the equivalent dual form condition:

$$\begin{array}{c} \exists\,(\xi^0,\eta^0) \in \mathbb{R}_+ \times \mathbb{R}^p_+ : \quad \eta^0_j\, g_j(x^0) = 0 \ \ (j = 1,\dots,p),\\ (\xi^0,\eta^0) \neq (0,\,0), \quad \xi^0 = 0 \ \text{ if } \ f'_-(x^0,u) = -\infty,\\ \eta^0_j = 0 \ \text{ if } \ {g_j}'_-(x^0,u) = -\infty \ \ (j = 1,\dots,p),\\ \text{and} \quad \xi^0 f'_-(x^0,u) + \sum_{j=1}^{p} \eta^0_j\, {g_j}'_-(x^0,u) \geq 0\,. \end{array} \tag{10}$$

Theorem 8.1 *Consider problem (7) and let x^0 be a radial minimizer. Let the functions g_j, $j = 1,\dots,p$, be radially continuous at x^0 when $j \notin J(x^0)$ and quasiconvex at x^0 when $j \in J(x^0)$ (or less restrictive, $g_j(x)$, $j \in J(x^0)$, jointly quasiconvex at x^0). Then for each $u \in X_0(x^0)$ condition (9) is satisfied.*

Theorem 8.2 *Consider problem (7). Let the functions g_j, $j \in J(x^0)$, be jointly strictly pseudoconvex at x^0, and f be pseudoconvex (strictly pseudoconvex) at x^0. Suppose also that when $(g_j)'_-(x^0,u) < 0$ holds for all $j \in J(x^0)$, then there is a sequence $t_k \to 0^+$ such that the following limits exist and satisfy the given inequalities $\lim_k \frac{1}{t_k}\left(g_j(x^0 + t_k u) - g_j(x^0)\right) < 0$. Suppose that for each $u \in X_0(x^0)$ condition (9) is satisfied. Then x^0 is a global minimizer (strict global minimizer).*

The next examples clarifies that the hypothesis for the radial continuity of g is essential in Theorem 8.1 (and hence in Theorem 5.3).

Example 8.1 Consider problem (8) with $f,\ g : \mathbb{R} \to \mathbb{R}$ given by $f(x) = -x$ and

$$g(x) = \begin{cases} -1\,, & x \leq 0\,,\\ \ \ 1\,, & x > 0\,. \end{cases}$$

The function g is quasiconvex and lower semicontinuous, but not (radially) continuous. The point $x^0 = 0$ is a radial (and global) minimizer. The Dini derivatives for $u = 1$ are $f'_-(x^0,u) = -1$ and $g'_-(x^0,u) = +\infty$. Condition (9) is not satisfied, since

$$(f'_-(x^0,u),\ g'_-(x^0,u)) = (-1,\ +\infty) \in -\overline{\mathrm{int}}\,(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}) = -\overline{\mathrm{int}}\,(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+[-g(x^0)])\,.$$

In connection with this example we do the following comment. The lower semicontinuity is in general a natural property in connection with minimization problems. The given example shows however that if we substitute the radial continuity hypothesis with lower semicontinuity hypothesis, we need add also some additional assumption in the necessary optimality conditions. We think this can be done on some abstract level (here we do not occupy with this problem). Concerning an eventual development in this direction, recall that a general approach to semi-continuous maps in cone-ordered spaces one finds in Penot, Théra [20].

The following example shows that the strict pseudoconvexity if g in Theorem 8.2 (and hence in Theorem 6.1) cannot be relaxed to only pseudoconvexity (or quasiconvexity).

Example 8.2 Consider problem (8) with $f,\ g:\mathbb{R}\to\mathbb{R}$ given by $f(x)=-x$ and $g(x)=0$. Put $x^0=0$. The functions f and g are pseudoconvex (and quasiconvex), but g is not strictly pseudoconvex. The point x^0 is not a global minimizer, while condition (9) is satisfied, since

$$(f'_-(x^0,u),\ g'_-(x^0,u))=(-u,\ 0)\notin-\overline{\text{int}}\,(\overline{\mathbb{R}}_+\times\overline{\mathbb{R}}_+)=-\overline{\text{int}}\,(\overline{\mathbb{R}}_+\times\overline{\mathbb{R}}_+[-g(x^0)])\,.$$

The following example shows that the pseudoconvexity requirement for f in Theorem 8.2 (and hence in Theorem 6.1) cannot be replaced by quasiconvexity.

Example 8.3 Consider problem (8) with $f,\ g:\mathbb{R}\to\mathbb{R}$ given by $f(x)=x^3$ and $g(x)=x$. Put $x^0=0$. The functions f and g are strictly quasiconvex, g is pseudoconvex at x^0 but f is not so. Since $f'_-(x^0,u)=0$ and $g'_-(x^0,u)=u$, condition (9) is satisfied (now $f'_-(x^0,u)\notin-\text{int}\,\overline{\mathbb{R}}_+$). However x^0 is not a global minimizer.

The conditions in dual form involve the pair (ξ^0,η^0) whose components can be referred as Lagrange multipliers. We see that in the considered here dual form conditions the multipliers depend on the direction. In contrary, classical optimization theory deals with directionally independent multipliers. The next example shows that the directional dependence of the multipliers for problems with continuous quasiconvex data cannot be avoided.

Example 8.4 Consider problem (8) with $f,\ g:\mathbb{R}\to\mathbb{R}$ given by

$$f(x)=\begin{cases} x\,,\ x\ge 0\,,\\ 2x\,,\ x<0\,,\end{cases}\qquad g(x)=\begin{cases} -2x\,,\ x\ge 0\,,\\ -x\,,\ x<0\,.\end{cases}$$

The functions f and g are continuous and strictly pseudoconvex (hence strictly quasiconvex). The set of the feasible points is $\mathbb{R}_+$. Put $x^0=0$. Obviously x^0 is a global minimizer. Condition (10) is satisfied in virtue of Theorem 8.1 (and Theorem 5.3), but a similar condition with directionally independent multipliers does not hold.

Indeed, assume in the contrary, that condition (10) is satisfied with some directionally independent multipliers (ξ^0,η^0). For $u\ge 0$ it holds $f'_-(x^0,u)=u$, $g'_-(x^0,u)=-2u$, whence in particular we should have

$$\xi^0\,f'_-(x^0,\ 1)+\eta^0\,g'_-(x^0,\ 1)=\xi^0-2\eta^0\ge 0\,.$$

Similarly, for $u\le 0$ it holds $f'_-(x^0,u)=2u$, $g'_-(x^0,u)=-u$, whence in particular we should have

$$\xi^0\,f'_-(x^0,\ -1)+\eta^0\,g'_-(x^0,\ -1)=-2\xi^0+\eta^0\ge 0\,.$$

Adding the two inequalities we obtain $-(\xi^0+\eta^0)\ge 0$, which obviously contradicts to $\xi^0\ge 0$, $\eta^0\ge 0$, $(\xi^0,\eta^0)\ne(0,\ 0)$.

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